

Necessary and Sufficient Condition for Nonsingular Fisher Information Matrix in ARMA Models

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Abstract

It is demonstrated that a necessary and sufficient condition that the Fisher information matrix of an ARMA model be nonsingular is that the model not be redundant, that is the autoregressive and moving-average polynomials do not share common roots.

Key words and phrases: ARMA model redundancy; Asymptotic covariance matrix of ARMA model.

The ARMA(p, q) model, which may be written in operator notation as, $\phi(B)z_t = \theta(B)a_t$, where $\phi(B) = 1 - \phi_1 B - \dots - \phi_p B^p$, $\theta(B) = 1 - \theta_1 B - \dots - \theta_q B^q$, a_t is white noise with variance $\sigma_a^2 > 0$ and B is the backshift operator on t , is said to be not redundant if and only if $\phi(B) = 0$ and $\theta(B) = 0$ have no common roots.

The Fisher information matrix may be written (Box and Jenkins, 1976, p.240), $I(\phi, \theta) = \sigma_a^{-2} E\{A_t A_t'\}$, where $A_t = (v_{t-1}, \dots, v_{t-p}, u_{t-1}, \dots, u_{t-q})$, $\phi(B)v_t = -a_t$, $\theta(B)u_t = a_t$ and A_t' denotes the transpose of A_t .

The inverse of this Fisher information matrix divided by the length of an observed time series can be used to obtain an estimate of the covariance matrix of the estimated parameters $\phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$. McLeod (1984) and Klein and Mélard (1990) have given algorithms for computing the inverse of the Fisher information matrix. It is frequently assumed without proof in the time series literature that the Fisher information matrix is nonsingular provided the ARMA model is not redundant. For examples see, Brockwell and Davis (1991, Theorem 8.11.1), and Hannan (1970, Theorem 8, p.403 and pp.413–414). Poskitt and Tremayne (1981, p.977) have proved this result but their approach is more complicated than that given below.

Theorem: The information matrix $I(\phi, \theta)$ is nonsingular if and only if the model is not redundant.

The proof of this theorem uses the following algebraic lemma.

Lemma: A necessary and sufficient condition for model redundancy is

that there exist nonzero polynomials $\alpha(B) = \prod_{j=1}^{p-1}(1 - E_j B)$, and $\beta(B) = \prod_{j=1}^{q-1}(1 - F_j B)$, such that

$$\alpha(B)\theta(B) = \beta(B)\phi(B). \quad (1.1)$$

Proof of Lemma : Let $\phi(B) = \prod_{j=1}^p(1 - G_j B)$, and $\theta(B) = \prod_{j=1}^q(1 - H_j B)$.

Then (1.1) can be written,

$$\prod_{i=1}^{p-1}(1 - E_i B) \prod_{j=1}^q(1 - H_j B) = \prod_{i=1}^{q-1}(1 - F_i B) \prod_{j=1}^p(1 - G_j B). \quad (1.2)$$

If $H_1 = G_1$ then (1.2) holds by setting $E_i = G_{i+1}$, ($i = 1, \dots, p-1$) and $F_i = H_{i+1}$, ($i = 1, \dots, q-1$). Conversely, if $G_i \neq H_j$, $\forall i, j$ then $\phi(B)/\theta(B)$ is irreducible and must be distinct from $\alpha(B)/\beta(B)$ since the degree of the polynomials occurring in the numerator and denominator is p and q in the latter irreducible case and $p-1$ and $q-1$ in the former.

Proof of Theorem: $I(\phi, \theta)$ is singular if and only if there exist $\alpha_0, \dots, \alpha_{p-1}$ and $\beta_0, \dots, \beta_{q-1}$, not all zero, such that

$$\text{Var}\left(\sum_{i=0}^{p-1} \alpha_i v_{t-i} + \sum_{j=0}^{q-1} \beta_j u_{t-j}\right) = 0 \quad (1.3)$$

where $\alpha_i \neq 0$ for some i and $\beta_j \neq 0$ for some j . This follows because $I(\phi, \theta) = \sigma_a^{-2} \mathbf{E}\{A_t A_t'\}$ is nonnegative definite. Eq. (1.3) is equivalent to the condition that there exists polynomials $\alpha(B) = \alpha_0 + \alpha_1 B + \dots + \alpha_{p-1} B^{p-1}$ and $\beta(B) = \beta_0 + \beta_1 B + \dots + \beta_{q-1} B^{q-1}$, at least one of which is nonzero, such that

$$\alpha(B)v_{t-1} + \beta(B)u_{t-1} = 0. \quad (1.4)$$

Eq. (1.4) can now be written

$$\frac{\alpha(B)}{\phi(B)}a_t - \frac{\beta(B)}{\theta(B)}a_t = 0. \quad (1.5)$$

Multiplying (1.5) by a_t and taking expected values, we see that $\alpha(B)/\phi(B) - \beta(B)/\theta(B) = 0$ which by the Lemma is a necessary and sufficient condition for model redundancy. This establishes that lack of model redundancy is a necessary and sufficient condition for the Fisher information matrix, $I(\phi, \theta)$ to be nonsingular.

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